Contents

1 Arnett's semi-analytic evolution

We follow A96 (which in turn follows Arnett `1979ApJ...230L..37`, `1980ApJ...237..541A`, `1982ApJ...253..785A`; a nice recent write-up for SN I – which have little effects od recombination – is in `2017ApJ...846...33A`) and consider a ball of gas with initial radius R_0 that is homologously expanding at constant velocity $v_{\rm sc}$, and has an initial thermal energy $\mathcal{E}_0$. The first law of thermodynamics can be written as,

$$\dot{E} + P\dot{V} = \epsilon M - L,$$

where E is the total energy, $V \equiv \frac{4\pi}{3}R^3$ is the volume, P the pressure, ϵ the energy generation rate (by radioactive decay) per unit mass, M the ejecta mass, and L the luminosity.

Assume the energy and pressure are dominated by radiation, i.e., $E \simeq \mathcal{E}$, and $P \simeq \frac{1}{3}\mathcal{E}/V$. Dividing by $\mathcal{E}$ on both sides and using homology, one finds,

$$4\frac{\dot{T}}{T} + 3\frac{\dot{R}}{R} + \frac{1}{3}3\frac{\dot{R}}{R} = 4\left(\frac{\dot{T}}{T} + \frac{\dot{R}}{R}\right) = \frac{1}{\tau_{\rm heat}} - \frac{1}{\tau_{\rm diff}},$$

where the heating timescale $\tau_{\rm h} = \mathcal{E}/\epsilon M$ and the diffusion timescale, which leads to a lumunosity $L_{\rm diff} = \mathcal{E}/\tau_{\rm diff}$, is also given by,

$$\tau_{\rm diff} = \frac{\kappa M}{\beta c R} = \tau_{\rm diff,0}\frac{R_0}{R},$$

where in the second equality one implicitly assumes constant opacity. For a constant density ball, $\beta = 13.8$.

We create a "one zone" model by assuming that the spatial ($x \equiv r/R$) and time dependence can be split,

$$T^4(x,t) = T_0^4\phi(t)\Psi(x).$$

For constant density $\rho = M/\frac{4}{3}\pi R^3$ and constant opacity κ,

$$\Psi(x) = \frac{\sin(\pi x)}{\pi x}.$$

In terms of these functions, the thermal energy can be written as,

$$\mathcal{E} = \int_0^R aT(r,t)^4 4\pi r^2\,dr = 4\pi R^3 aT(0,t)^4 \int_0^1 \Psi(x)x^2\,dx = \frac{4}{\pi}R_0^3 aT_0^4 \frac{R_0}{R}\phi(t),$$

where we used that $\int_0^1 \Psi(x)x^2\,dx = 1/\pi^2$. The factor R_0/R accounts for adiabatic expansion and $\phi(t)$ for radiation loss and radioactive heating. Generally, given the assumption of homology, we can write in terms of initial properties,

$$E = \mathcal{E}_0 \frac{R_0}{R}\phi(t).$$

Then, the general expression for the luminosity is,

$$L = \frac{\mathcal{E}}{\tau_{\rm diff}} = \frac{\mathcal{E}_0 \frac{R_0}{R}\phi(t)}{\tau_{\rm diff,0}\frac{R_0}{R}} = L_0\phi(t).$$

Supposing the initial thermal energy is of order the kinetic energy, i.e., $\mathcal{E}_0 \simeq \frac{1}{2}Mv_{\rm sc}^2$, the initial lumonosity $L_0 = \mathcal{E}_0/\tau_{\rm diff,0} \propto v_{\rm sc}^2 R/\kappa$ is independent of mass, but proportional to radius. Faster ejections (larger energy) from larger stars (faster diffusion) give more luminous transients.

1.1 Diffusion and heating

With just diffusion and heating, one has

$$\frac{\frac{d}{dt}(TR)^4}{(TR)^4} = \frac{\dot\phi}{\phi} = \frac{1}{\tau_{\rm heat}} - \frac{1}{\tau_{\rm diff}} \qquad \Leftrightarrow \qquad \frac{\dot\phi}{\phi} = \left[\frac{\epsilon/\epsilon_0}{\tau_{\rm heat,0}\phi} - \frac{1}{\tau_{\rm diff,0}}\right]\frac{R}{R_0},$$

where we tried to write in terms of ratios on the right-hand side, with ϵ/ϵ_0 capturing the time dependence of the heating process (and where we again implicitly assumed constant opacity).

Ignoring heating, an analytic solution is possible. Using that $\tau_{\rm diff} = \tau_{\rm diff,0}(R_0/R) = \tau_{\rm diff,0}/(1 + v_{\rm sc}t/R_0)$, and defining an expansion timescale $\tau_{\rm exp,0} = R_0/v_{\rm sc}$, one finds

$$\phi = \exp\left(-\frac{t}{\tau_{\rm diff,0}} - \frac{t^2}{2\tau_{\rm exp,0}\tau_{\rm diff,0}}\right).$$

Generally, $\tau_{\rm diff,0} \gg \tau_{\rm exp,0}$, and thus for $t > \tau_{\rm exp,0}$, the lightcurve is essentially a Gaussian, with a timescale that is the geometric mean of the expansion and diffusion times scales, $\tau_{\rm lc} = \sqrt{\tau_{\rm exp,0}\tau_{\rm diff,0}} \propto \sqrt{\kappa M/v_{\rm sc}}$. Slower, more massive ejections lead to longer transients.

Including heating, the integration needs to be done numerically. However, generally, one expects maximum to occur when $\dot{\phi} = 0$, i.e., when $1/\tau_{\rm heat} = 1/\tau_{\rm diff}$ (of course, if heating is too small, this maximum after explosion never happens). From their definitions, the timescales match when $L = \epsilon M$. Thus, maximum luminosity gives a measure of the total amount of radioactive decay – "Arnett's rule." (This will be an underestimate if the opacity is decreasing with time – or if this is happening effectively due to recombination, but an overestimate if the heat is deposited deep down and thus one sees heat diffusing out from a time that there was more decay.)

1.2 Including recombination

At some temperature T_i, material will recombine and become essentially transparent. If this happens inside the cloud, then this will effectively be at optical depth zero, and the photosphere would be at $T_{\rm eff}^4 \simeq 2T_i^4$. As more matter recombines, the photosphere will move in, with recombination and advection ("freed" radiation) giving additional sources of luminosity. At this time, one will have,

$$L_{\rm diff} + L_{\rm adv} + L_{\rm rec} = L_{\rm min} = 4\pi R_i^2 \sigma 2 T_i^4,$$

where $R_i = x_i R$ is the radius of the recombination front, and where we used the subscript "min" as a reminder that the luminosity cannot be lower than this value for this radius.

The luminosity due to recombination is

$$L_{\rm rec} = -4\pi R_i^2 \dot{R}_i \rho Q = -3x_i^2 \dot{x}_i \frac{4\pi}{3} R^3 \rho Q = -3x_i^2 \dot{x}_i M Q,$$

where Q is the energy release per unit mass due to recombination.

For the advection and diffusion terms, the results depend on whether the front moves slow or fast compared to the time to adjust the overall temperature structure. Generally, though, $L_{\rm diff} = \mathcal{E}/\tau_{\rm diff}$ and,

$$L_{\rm adv} = -\dot{x}_i \frac{\partial \mathcal{E}}{\partial x_i}$$

but the total thermal energy $\mathcal{E}$ and diffusion timescale $\tau_{\rm diff}$ may now depend on x_i. In consequence, not only the differential equation for ϕ has to be solved, but also one for the recombination front position x_i. The latter can be derived from the constraint that the additional luminosity $L_{\rm rec}+L_{\rm adv}$ has to match the excess luminosity $L_{\rm min}-L_{\rm diff}$, or

$$-\dot{x}_i\left[3x_i^2MQ+\frac{\partial\mathcal{E}}{\partial x_i}\right]=4\pi R^2x_i^22\sigma T_i^4-\frac{\mathcal{E}}{\tau_{\rm diff}}$$

Below, we will also use the timescale on which the initial energy would be radiated at an effective temperature of $2^{1/4}T_i$,

$$\tau_{\rm i,0}\equiv\frac{\mathcal{E}_0}{L_{\rm min,0}}=\frac{\frac{4}{\pi}R_0^3aT_0^4}{4\pi R_0^22\frac{ac}{4}T_i^4}=\frac{4R_0}{\pi^2c}\frac{T_0^4}{2T_i^4}.$$

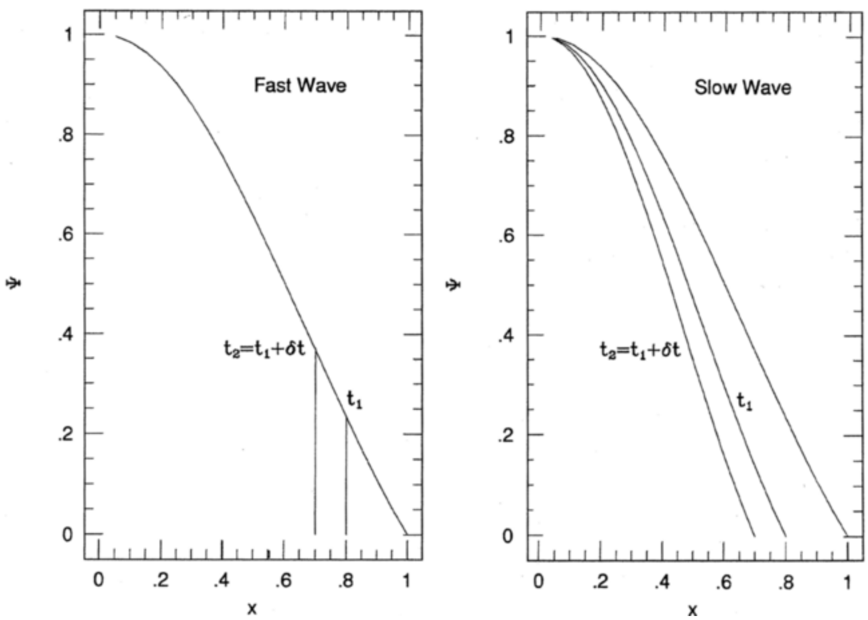

Figure 1: Fast and slow approximation to a recombination wave. From A96, his Fig. 13.7.

1.2.1 Slow recombination front

If the recombination front moves slowly, photon diffusion inside it will ensure the temperature structure adjusts to its new outer boundary, $R_i=x_iR$, with the same spatial structure ($[T(x)/T(0)]^4=\Psi(x/x_i)$). Thus, the total thermal energy will be

$$\mathcal{E} = 4\pi R^3 a T(0,t)^4 \int_0^{x_i} \Psi(x/x_i) x^2 dx = \mathcal{E}_0 \frac{R_0}{R} \phi(t) x_i^3,$$

where $\phi(t)$ accounts for changes in central properties due to the recombination wave and associated energy loss. Given this, the advection luminosity is given by,

$$L_{\rm adv} = -\dot{x}_i \frac{\partial \mathcal{E}}{\partial x_i} = -3x_i^2 \dot{x}_i \mathcal{E}_0 \frac{R_0}{R} \phi(t).$$

Since the size is decreasing, the luminosity due to photon diffusion also changes, becoming

$$L_{\rm diff} = \frac{\mathcal{E}}{\tau_{\rm diff}} = \frac{\mathcal{E}_0}{\tau_{\rm diff,0}} \phi(t) x_i,$$

where we used that $\tau_{\rm diff} = \tau_{\rm diff,0}(R_0/R)x_i^2$, with the dependence on x_i^2 reflecting the dependence of $\tau_{\rm diff}$ on M/R (for constant density the mass enclosed within the recombination front scales with x_i^3). The differential equations to be solved thus become,

$$\frac{\dot{\phi}}{\phi} = \frac{\epsilon M}{\mathcal{E}_0 \phi x_i^3} \frac{R}{R_0} - \frac{1}{\tau_{\rm diff,0} x_i^2} \frac{R}{R_0},$$

$$-3x_i^2 \dot{x}_i \left[MQ + \mathcal{E}_0 \frac{R_0}{R} \phi \right] = 4\pi R^2 x_i^2 2\sigma T_i^4 - \frac{\mathcal{E}_0}{\tau_{\rm diff,0}} \phi x_i.$$

Simplifying,

$$\frac{\dot{\phi}}{\phi} = \left[\frac{\epsilon/\epsilon_0}{\tau_{\rm heat,0} \phi x_i^3} - \frac{1}{\tau_{\rm diff,0} x_i^2} \right] \frac{R}{R_0},$$

$$-3x_i^2 \dot{x}_i = \frac{\dfrac{x_i^2}{\tau_{\rm i,0}} \left(\dfrac{R}{R_0} \right)^2 - \dfrac{\phi x_i}{\tau_{\rm diff,0}}}{\dfrac{MQ}{\mathcal{E}_0} + \dfrac{R_0}{R} \phi}$$

1.2.2 Fast recombination front

For a fast-moving recombination front, the temperature structure inside will not react to the fact that the outer parts are being chopped off. The total thermal energy inside the recombination wave is,

$$\mathcal{E}_{x<x_i} = 4\pi R^3 a T(0,t)^4 \int_0^{x_i} \Psi(x) x^2 \, dx = \mathcal{E}_0 \frac{R_0}{R} \phi(t) \, \pi^2 \int_0^{x_i} \Psi(x) x^2 \, dx.$$

Using that $d/dx_i \int_0^{x_i} \psi(x)x^2 dx = x_i^2\psi(x_i)$, the advection luminosity is given by,

$$L_{\rm adv} = -\dot{x}_i \frac{\partial \mathcal{E}_{x<x_i}}{\partial x_i} = -3x_i^2\dot{x}_i\frac{\pi^2}{3}\Psi(x_i)\mathcal{E}_0\frac{R_0}{R}\phi(t).$$

The luminosity due to photon diffusion from the inside now changes only because we are evaluating it at a different position, becoming

$$L_{\rm diff} = L^0_{\rm diff}\phi(t)\frac{\left|-x^2\partial\Psi/\partial x\right|_{x_i}}{\left|-x^2\partial\Psi/\partial x\right|_1} = \frac{\mathcal{E}_0}{\tau_{\rm diff,0}}\phi(t)\left|-x^2\frac{\partial\Psi}{\partial x}\right|_{x_i} = \frac{\mathcal{E}_0}{\tau_{\rm diff,0}}\phi(t)\pi^2 I(x_i).$$

where $L^0_{\rm diff}$ is the diffusion luminosity we would obtain ignoring the recombination wave, and where we have used that $[-x^2\partial\Psi/\partial x]_{x_i} = (1/\pi)\sin(\pi x_i) - x_i\cos(\pi x_i) = \pi^2 I(x_i)$ (where $\pi^2 I(x_i) = \pi^2\int_0^{x_i}\Psi(x)x^2dx$ is the normalised integral).

The differential equations to be solved now become,

$$\frac{\dot{\phi}}{\phi} = \frac{\epsilon M}{\mathcal{E}_0\phi\pi^2 I(x_i)}\frac{R}{R_0} - \frac{1}{\tau_{\rm diff,0}}\frac{R}{R_0},$$

$$-3x_i^2\dot{x}_i\left[MQ + \mathcal{E}_0\frac{R_0}{R}\phi\frac{\pi^2}{3}\Psi(x_i)\right] = 4\pi R^2x_i^2 2\sigma T_i^4 - \frac{\mathcal{E}_0}{\tau_{\rm diff,0}}\phi\pi^2 I(x_i).$$

Simplifying,

$$\frac{\dot{\phi}}{\phi} = \left[\frac{\epsilon/\epsilon_0}{\tau_{\rm heat,0}\phi\pi^2 I(x_i)} - \frac{1}{\tau_{\rm diff,0}}\right]\frac{R}{R_0},$$

$$-3x_i^2\dot{x}_i = \frac{\dfrac{x_i^2}{\tau_{\rm i,0}}\left(\dfrac{R}{R_0}\right)^2 - \dfrac{\phi\pi^2 I(x_i)}{\tau_{\rm diff,0}}}{\dfrac{MQ}{\mathcal{E}_0} + \dfrac{R_0}{R}\phi\dfrac{\pi^2}{3}\Psi(x_i)}$$

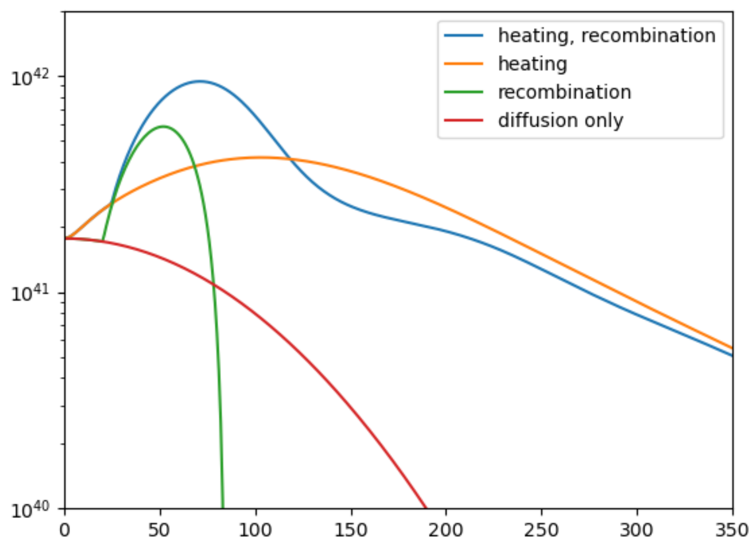

Figure 2: Comparison of explosions with and without recombination and heating by radioactive decay. Shown are curves for the "fast" since for the "slow" case I could not reproduce the curves in A96 as well. Still, the general trends are clear and should be correct.

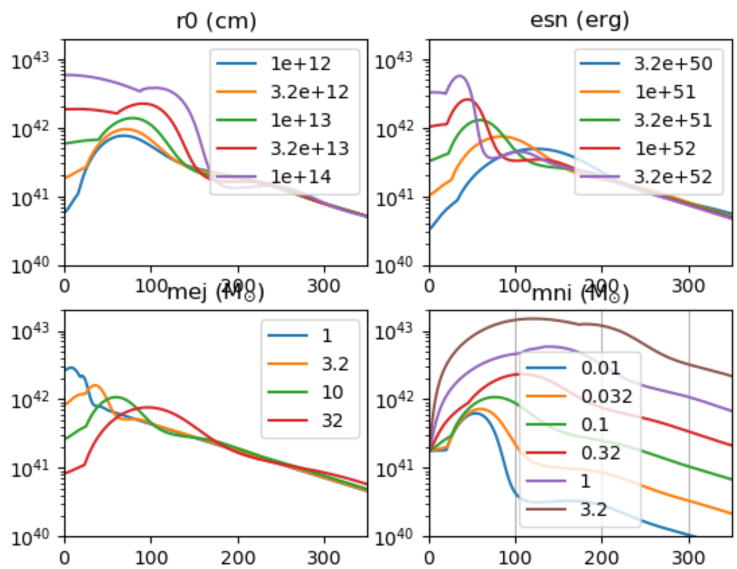

Figure 3: Semi-analytic lightcurves for supernovae with varying properties. Those not varied are held fixed at those inferred for SN 1987A by A96 (his Table 13.2): $M_{\rm ej} = 15\,M_\odot$, $E_{\rm SN} = 1.7\,{\rm B}$, $R_0 = 3\times10^{12}\,{\rm cm}$, $\kappa = 0.2\,{\rm cm}^2\,{\rm g}^{-1}$, $M_{\rm Ni} = 0.075\,M_\odot$, $T_{\rm ion} = 4500\,{\rm K}$, $Q_{\rm ion} = 13.6\,{\rm eV\,nucleon}^{-1}$. Recombination uses the "fast" prescription. Ignored are losses of gamma rays, and hence the luminosity at late times is overestimated.